Announcements

1) Math Club talk Wed
3-4, CB 2063

Student presentation
- Math Modeling Competition

2) New (last) HW up
on CTools this afternoon

Chapter 5

Differentiation.

Only for $f: \mathbb{R} \to \mathbb{R}$ . . . for now!

Definition: Let $f: \mathbb{R} \to \mathbb{R}$.

f is differentiable at $x = a$

if $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists.

If the limit exists, we set

$$\boxed{f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$

and we call f' the derivative of f at $x = a$.

Remark: We can replace the domain of f by an interval, but (for now) not by an arbitrary set.

If the interval contains endpoints, we need to take care, since then the limit defining $f'(a)$ may be only one-sided.

Proposition: Let f be differentiable at $x=a$. Then f is continuous at $x=a$.

proof: Suppose

$$f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

exists.

Write $f(x)-f(a) = \dfrac{f(x)-f(a)}{x-a} \cdot x-a$

Then taking limits,

$$\lim_{x \to a} \left(f(x) - f(a) \right) = \lim_{x \to a} \left(\frac{f(x)-f(a)}{x-a} \cdot (x-a) \right)$$

$$= \lim_{x \to a} \frac{f(x)-f(a)}{x-a} \cdot \lim_{x \to a} (x-a)$$

Since both limits exist!

$$= f'(a) \cdot 0 = 0.$$

Hence, $\lim_{x \to a} (f(x) - f(a)) = 0$

But $\lim_{x \to a} (f(x) - f(a))$

$$= \left(\lim_{x \to a} f(x) \right) - f(a)$$

This shows us that

$$\lim_{x \to a} f(x) = f(a),$$

so f is continuous

at $x = a$. $\square$

But there are easy

counterexamples to the

converse!

Example 1: ($f(x)=|x|$)

Consider $a=0$.

$$\lim_{x\to 0}|x| = 0 = f(0)$$

so f is continuous at $a=0$

However, if $\boxed{x>0}$, then

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{|x|}{x}$$

$$= \lim_{x\to 0}\frac{x}{x}$$

$$= 1$$

But if $\boxed{x < 0}$,

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{|x|}{x}$$

$$= \lim_{x \to 0} \frac{-x}{x}$$

$$= -1.$$

Therefore, $f(x) = |x|$ is

not differentiable at $x = 0$,

even though it is continuous

Question: Can we find a continuous function that fails to be differentiable on an arbitrary subset of $\mathbb{R}$?

Moreover, can we find a continuous function that is nowhere differentiable on $\mathbb{R}$? Yes! Later...

Proposition: Let f and g be differentiable at $x=a$. Then if $c\in\mathbb{R}$,

1) $(cf)'(a) = c\cdot f'(a)$

2) $(f\pm g)'(a) = f'(a)\pm g'(a)$

3) $(f\cdot g)'(a) = f'(a)g(a)+g'(a)f(a)$ (product rule)

4) $\left(\frac{f}{g}\right)'(a) = \dfrac{g(a)f'(a)-f(a)g'(a)}{(g(a))^2}$

if $g(a)\neq 0$ (quotient rule)

proof: a) trivial.

b) $$\lim_{x\to a} \frac{\left(f(x)+g(x)\right)-\left(f(a)+g(a)\right)}{x-a}$$

$$= \lim_{x\to a} \frac{f(x)-f(a)+g(x)-g(a)}{x-a}$$

$$= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} + \lim_{x\to a} \frac{g(x)-g(a)}{x-a}$$

since both limits exist

$$= f'(a) + g'(a).$$

Replace "+" by "−" for
$(f-g)'(a) = f'(a) - g'(a)$

2) A little nontrivial

$$\lim_{x \to a} \frac{f(x)g(x) - f(a)g(a)}{x-a}$$

$$= \lim_{x \to a} \frac{f(x)g(x) \overbrace{- f(a)g(x) + f(a)g(x)}^{=0} - f(a)g(a)}{x-a}$$

$$= \lim_{x \to a} \frac{f(x)g(x) - f(a)g(x)}{x-a} + \lim_{x \to a} \frac{f(a)g(x) - f(a)g(a)}{x-a}$$

Since both limits exist

because

$$\lim_{x \to a} \frac{f(a)g(x) - f(a)g(a)}{x-a}$$

$$= f(a) \lim_{x \to a} \frac{g(x)-g(a)}{x-a}$$

constant

$$= f(a) \cdot g'(a)$$

and

$$\lim_{x \to a} \frac{f(x)g(x) - f(a)g(x)}{x-a}$$

$$= \lim_{x \to a} \left(\frac{f(x)-f(a)}{x-a} \right) g(x)$$

$$= \lim_{x \to a} \frac{f(x)-f(a)}{x-a} \cdot \lim_{x \to a} g(x)$$

$$= f'(a) \cdot g(a)$$ since g is continuous at $x=a$.

This shows

$$(f \cdot g)'(a) = f'(a)g(a) + g'(a)f(a)$$

d) Is a consequence of

the product rule plus . .

Chain Rule: If g is differentiable at $x=a$ and f is differentiable at $x=g(a)$, then

$$\boxed{(f\circ g)'(a) = f'(g(a))\cdot g'(a)}$$

fake proof:

$$\lim_{x\to a} \frac{f(g(x))-f(g(a))}{x-a}$$

$$= \lim_{x\to a} \frac{f(g(x))-f(g(a))}{g(x)-g(a)} \cdot \frac{g(x)-g(a)}{x-a}$$

$$= f'(g(a))\,g'(a)$$ done! Fake!

Theorem: (Intermediate value)

Theorem: (Max/Min)

Corollary: (Int. Value Theorem)

Note. Conway's example!

List of Cool Things

1) Cantor set uncountable
   (bijection w/ ternary expansion)

2) Set of discontinuities

3) Conway's IVT converse

4) Rearrangements of Infinite series

More?